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International Journal of Multiphase Flow 25 (1999) 921–941

International Journal of Multiphase Flow

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Numerical study of the oscillations of a non-spherical bubble in an inviscid, incompressible liquid. Part II: the response to an impulsive decrease in pressure

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Received 24 March 1998; received in revised form 6 November 1998

Abstract

We consider the response of a gas bubble in an inviscid, incompressible fluid, to a rapid, impulsive change of the ambient pressure, from some initial value down to a minimum and then back up to the initial value. The most important result is the sharp *decrease* in the amplitude of volume oscillations which results from finite amplitude coupling between the purely radial and shape modes. When the bubble has a steady shape that is non-spherical, there can be resonance between radial mode oscillations and one of the shape modes. This contributes to growth of the resonant shape mode from an initial spherical shape, and produces a 'phase-lock' with the two modes either in-phase or π radians out-of-phase. For in-phase conditions, this leads ultimately to a geometric amplification of higher-order shape modes and a sharp decrease in the amplitude of the volume oscillations. The same geometric amplification mechanism also appears in the absence of mean shape deformation (and thus 1:1 resonance), but only if there is another mechanism for the appearance of shape modes. There is also no phase-locking when there is no mean deformation, and so the importance of the geometric amplification effect is greatest whenever the phase difference between P_0 and P_2 is small. © 1999 Published by Elsevier Science Ltd. All rights reserved.

1. Introduction

As part of a continuing effort to understand the effect of interactions between oscillations of volume and shape on the observable dynamics of an oscillating bubble, we have conducted a numerical investigation of finite amplitude bubble oscillations in an inviscid liquid due to an impulsive decrease in pressure. This work builds upon our previous study of non-spherical

PII: S0301-9322(98)00074-3

bubble dynamics involving the free oscillations of a bubble in an inviscid, quiescent fluid, Part I. In moving to an impulsive decrease in pressure, we take one step toward modeling physically realizable systems.

The recent experimental results of Ceccio and Brennen (1991) clearly show that cavitation bubbles on blunt headforms are non-spherical and emit an acoustic impulse that is significantly smaller than that predicted by Rayleigh–Plesset theory. As cavitation bubbles move along streamlines near the blunt headform, they pass through a region of low pressure causing them to first grow and then to collapse or oscillate producing sound. It has been suggested that the non-spherical shape of the bubble 'defocuses' the bubble collapse making the bubble a less efficient source of sound.

Previous work on resonant interactions between volume and shape oscillations has shown that energy can be transferred away from volume oscillations and into shape oscillations when the natural frequency for volume oscillations is close to an integer multiple of the natural frequency for shape oscillations. Yang et al. (1993) and Feng and Leal (1993) have shown that the 1:1 resonant interaction is significant when there is a mean deformation due to an external flow or anisotropic pressure field. Having these things in mind we undertook a numerical investigation to determine whether the 1:1 resonance interaction for finite amplitude oscillations could be responsible for the observed difference between the predictions of Rayleigh–Plesset theory and the experimental results of Ceccio and Brennen (1991).

In the course of our investigation we discovered a non-resonant finite amplitude interaction between volume and shape oscillations. As the non-spherical bubbles collapse, energy is transferred into *all* of the shape modes present in the bubble shape, which are thus amplified. This mechanism is similar to the effect suggested many years ago by Birkhoff (1954, 1956) in which collapsing bubbles are predicted to be unstable even though the heavier fluid is being accelerated toward the lighter fluid, a situation which Taylor (1950) found to be stable.

In the present work, we present numerical simulations of bubbles with a mean deformation of shape that experience an impulsive decrease in pressure in an attempt to study the physics underlying the observations of Ceccio and Brennen (1991). We find that, in addition to the 1:1 resonant interaction between oscillations of volume and shape, the non-resonant interaction between volume and shape mentioned above reduces the efficiency of the bubble as a monopole source of sound. We demonstrate that the non-resonant interaction *does not* require the presence of a mean deformation, but that the presence of higher modes in the interface is important.

2. The numerical procedures

The numerical results reported in this paper were obtained using the boundary-integral technique that is described in Part I. For this study of problems with an impulsive pressure decrease, we have found it necessary to periodically redistribute the nodes along the interface as the bubble expands and collapses (Boulton-Stone, 1993; Oguz and Prosperetti, 1990). This is accomplished by placing new nodes along the interface at equal intervals of the length and assigning to them the linearly interpolated value from the original nodes. The ability of the method, including the redistribution of nodes along the interface, to reproduce the predictions

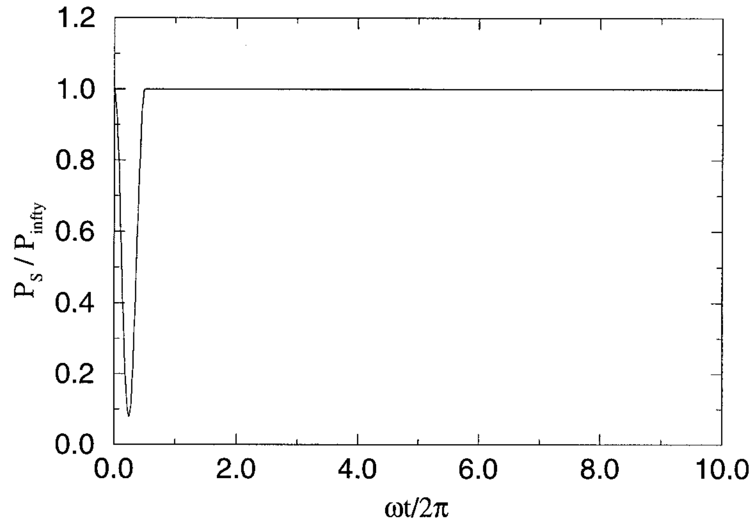

Fig. 1. The isotropic pressure history (P_S/P_∞) versus $\omega_0 t/2\pi$) for $\Delta_P = 0.92$.

of Rayleigh–Plesset theory as well as its convergence for increasing number of nodes and decreasing time-step size can be found in McDougald (1997).

In order to see the interaction between modes, we use the Legendre polynomial decomposition described in Part I to extract the amplitudes of the radial and shape modes from the bubble shapes calculated using the boundary integral method. We present the results of our simulations as the amplitudes of the radial, P_0, mode and shape, P_n, modes as a function of time non-dimensionalized using the natural frequency of the radial mode $\omega_0/2\pi$. In addition, we compare the volume response and acoustic pressure response with the predictions of Rayleigh–Plesset theory for several cases.

The steady-state shape for the bubble in the following results is made non-spherical by a non-uniform pressure distribution imposed on the interface, $A(\eta,t) = A_0(t) + A_n(t)P_n(\eta)$ as outlined in Part I, Section 2. The amplitudes of the radial and shape modes at steady-state along with values of the pressure coefficients are given in Table 1. This case corresponds to case G from Section 4.2 of Part I. As noted there, the mean bubble shape is oblate, with a half length along the symmetry axis of 0.86 and perpendicular to the symmetry axis of 1.07. By including this modest mean deformation we hope to at least qualitatively capture the physics associated with the larger mean deformations of shape observed in experimental systems. We will see later, however, that the finite amplitude effect that is primarily responsible for

Table 1
The equilibrium values of $\epsilon\alpha_{1,0}$, $\epsilon\alpha_{1,2}$ and the corresponding values of A_0 and A_2

$\epsilon\alpha^{SS}_{1,0}$	$\epsilon\alpha^{SS}_{1,2}$	A_0	A_2
0.00	-0.14	$O(10^{-3})$	0.61

decreasing the acoustic response of a bubble *does not* depend upon the presence of a mean deformation.

The bubble oscillations are initiated by an impulsive decrease in the isotropic pressure at the bubble surface such that

$$\frac{P_S}{P_\infty} = 1 - \Delta_P \frac{(1 - \cos 2\omega_0 t)}{2} \quad 0 \le t \le \pi/\omega_0$$

$$\frac{P_S}{P_\infty} = 1 \quad t > \pi/\omega_0 \tag{1}$$

where P_S is the pressure at the bubble surface and Δ_P is the magnitude of the pressure impulse. The pulse width depends upon the natural frequency of the radial mode so that results for various amounts of detuning can be compared. The pulse width is constant with respect to $\omega_0 t/2\pi$ and is equal to one half of the period of the radial mode oscillations. A representative isotropic pressure history is presented in Fig. 1. The impulse height is adjusted so that radial mode amplitude always grows to the same value during the initial expansion of the bubble. Rather than use trial and error with the boundary integral code to find the appropriate impulse height, we use the Rayleigh–Plesset theory for this purpose.

3. Results

First, we present a series of results in the neighborhood of the 1:1 resonance condition where we anticipate coupling between radial and shape oscillations based upon the small deformation theory and the numerical investigation of Part I. In this sequence of results, the equilibrium conditions are varied to give different degrees of mismatch between the natural frequencies for radial and shape mode oscillations, including the frequency shift associated with the mean shape deformation due to $A_2 \neq 0$. In particular, the detuning parameter, $\epsilon\beta^*$, for this case is given by

$$\epsilon\beta^* = \epsilon\beta_0^* + \left(\frac{3\gamma - 1}{2\omega_n} + \frac{3\gamma - 1}{\omega_n^3}\right)A_0 - \frac{(n+1)(2n+1)k_n}{(3n+1)\omega_n}\left(\frac{2(n+1)}{\omega_n^2} + \frac{10-n}{4}\right)A_n$$

$$k_n = \left(\frac{(n-1)!!}{(n/2)!}\right)^3 \frac{(3n/2)!}{(3n-1)!!} \quad n \text{ even}$$

$$k_n = 0 \quad n \text{ odd} \tag{2}$$

as shown in Part I. Here $\epsilon\beta_0^*$ is the difference $\omega_0 - \omega_n$ for a spherical bubble, and $\epsilon\beta^*$ is the same quantity corrected to account for the shifts in natural frequency due to nonzero deformation, i.e. $A_0, A_n \neq 0$, as predicted by small deformation theory. As the detuning parameter is decreased, from positive values, the natural frequency for radial mode oscillations is decreased relative to the natural frequency for P_2 shape mode oscillations. If we recall the definitions of ω_0 and ω_n from Part I, we see that decreasing the difference between ω_0 and ω_n corresponds to decreasing the surface tension or increasing the equilibrium radius by lowering the ambient pressure. At some point the natural frequencies will become equal, the conditions for 1:1 resonance will be satisfied, and the coupling between radial and P_2 mode oscillations will be maximal. The small deformation theory predicts that the greatest interaction between the radial mode and the P_2 shape mode will occur when the effective detuning parameter $\epsilon\beta^* = 0$. However, the numerical work on case G in Part I shows that the maximum interaction actually occurs at $\epsilon\beta^* \approx -0.21$ due to differences between the finite amplitude, numerically

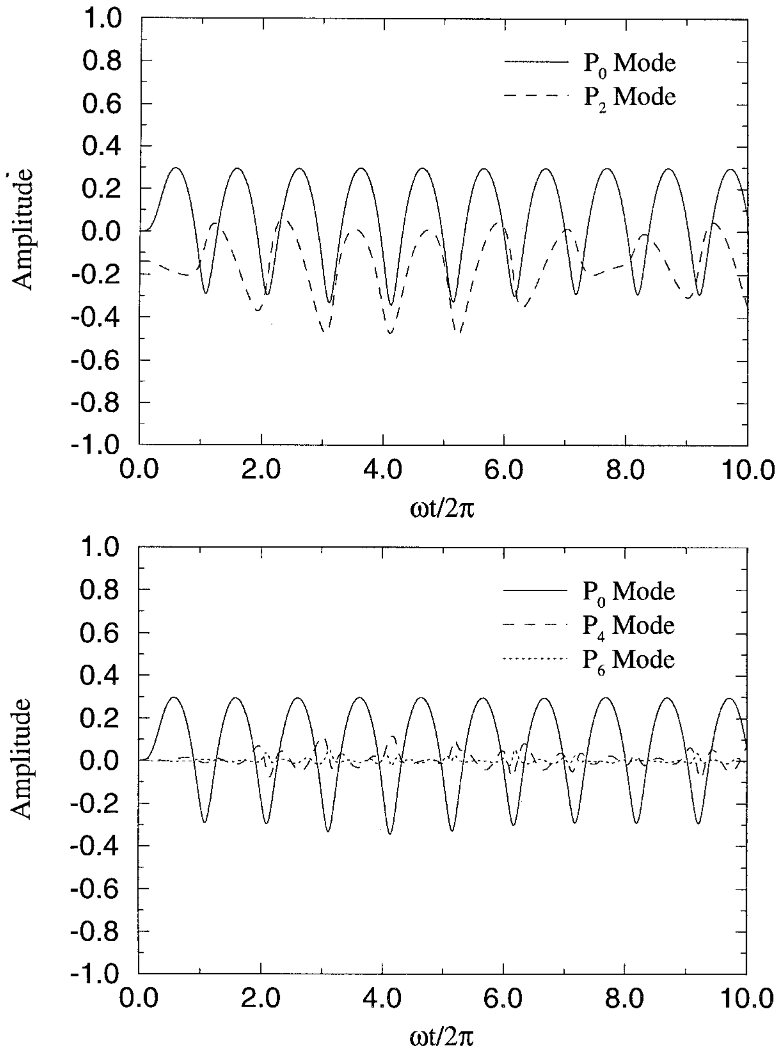

Fig. 2. Amplitudes of the P_0 and P_2 modes (top) and the P_0, P_4 and P_6 modes (bottom) for $\epsilon\beta^* = 0.29$.

determined shift in the natural frequency of the P_2 mode (caused by the mean shape deformation) and the predicted, small deformation value (see Section 4.2, Part I). In both the small deformation theory and the earlier numerical study, the coupling between modes decreases as the equilibrium conditions are changed and the mismatch between the natural frequencies of the radial mode and shape mode increases. Significant departures from the Rayleigh–Plesset theory appear for conditions where the detuning parameter $\epsilon\beta^*$ falls within a range of values centered on the value exhibiting the greatest interaction between modes. In the present study, we observe that the finite amplitude interaction between radial and shape oscillations is *not* symmetric with regard to the detuning parameter $\epsilon\beta^*$ corresponding to the maximum interaction as predicted by either the small deformation theory or the free oscillation finite amplitude numerical results of Part I.

The impulse height Δ_P for the series of results shown in Figs. 2–14 is adjusted so that the radial mode grows to $\epsilon\alpha_{1,0}=0.30$ during the initial expansion of the bubble. In Fig. 2, where the detuning parameter $\epsilon\beta^*=0.29$ there is very little interaction between the radial mode and shape modes. In fact, the effect of the coupling between modes on the amplitude of the radial mode is almost imperceptible and the volume response is equivalent to that predicted by Rayleigh–Plesset theory, as shown in Fig. 3. However, small amplitude oscillations of both the P_2 and P_4 shape modes appear in the bubble's response to an impulsive pressure decrease.

As the equilibrium conditions are changed so that the detuning parameter $\epsilon\beta^*$ decreases to $\epsilon\beta^*=0.09$, the interaction between modes increases and the amplitudes of the shape mode oscillations reach higher values. In Fig. 4, a correlation between the collapse of the bubble and the maximum amplitude of the P_4 mode oscillations appears briefly around $\omega_0 t/2\pi=5$. This is the first appearance of *non-resonant* large amplitude interactions between oscillations of the radial mode and the shape modes.

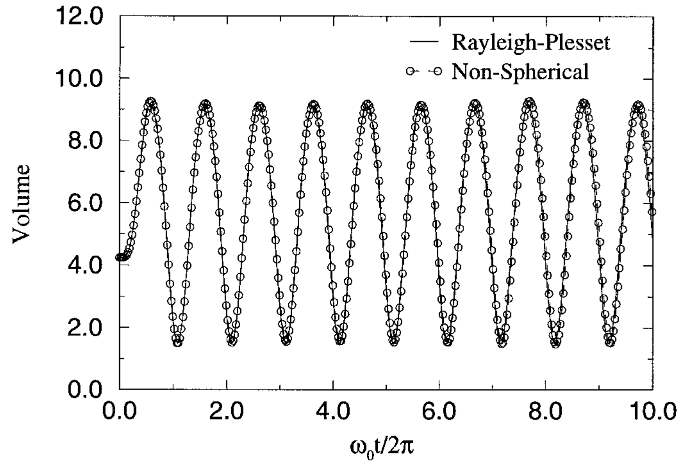

Fig. 3. Comparison between the volume response predicted from the Rayleigh–Plesset theory and numerical simulation of a non-spherical bubble for $\epsilon\beta^*=0.29$.

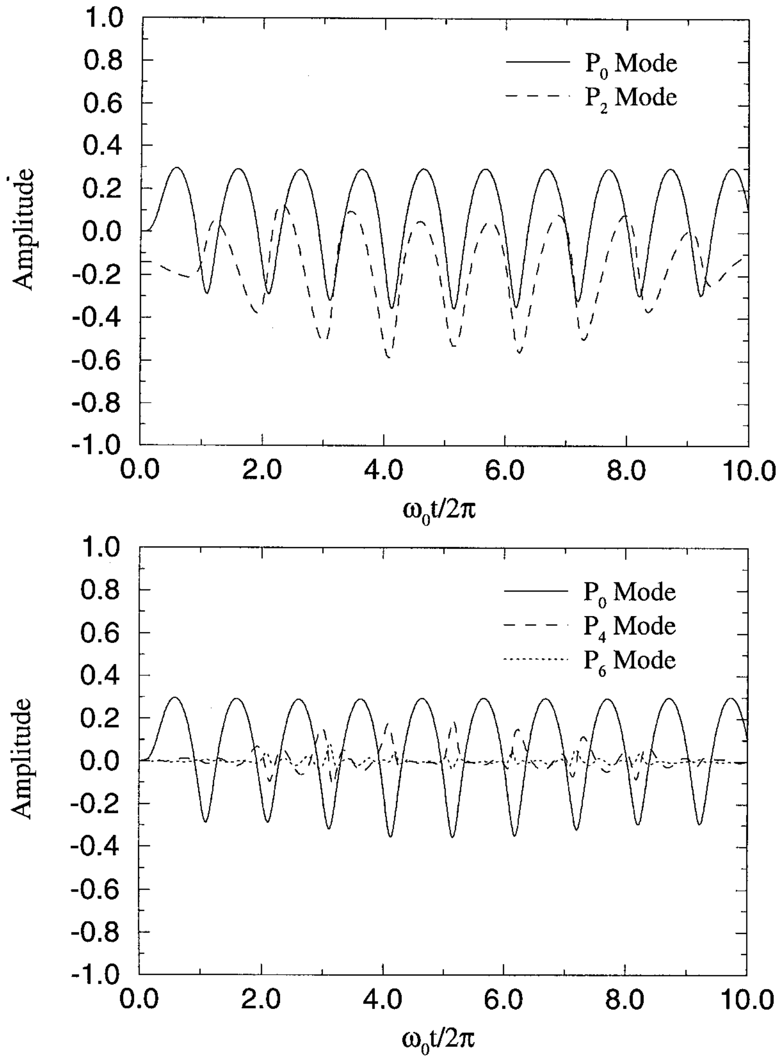

Fig. 4. Amplitudes of the P_0 and P_2 modes (top) and the P_0, P_4 and P_6 modes (bottom) for $\epsilon\beta^* = 0.09$.

As conditions are changed to bring the natural frequencies for radial and P_2 shape mode oscillations closer together, the interaction between the modes continues to become stronger. The result for $\epsilon\beta^* = -0.21$, shown in Fig. 5, corresponds to the $\epsilon\beta^*$ value that led to the greatest interaction between the radial and P_2 mode in the study of free oscillations in Part I. Initially, the P_2 shape mode grows via 1:1 resonant coupling with the radial mode and the higher modes then become excited via non-resonant modal coupling with the P_2 mode as observed in Part I. However, in this case, the large amplitude radial oscillations also interact *directly* with the higher-order shape modes once they become excited. The collapse of bubble volume which occurs at the minimum of the P_0 mode is accompanied by a direct amplification of the P_4 and P_6 modes. It will be noted that the interaction between modes is still time

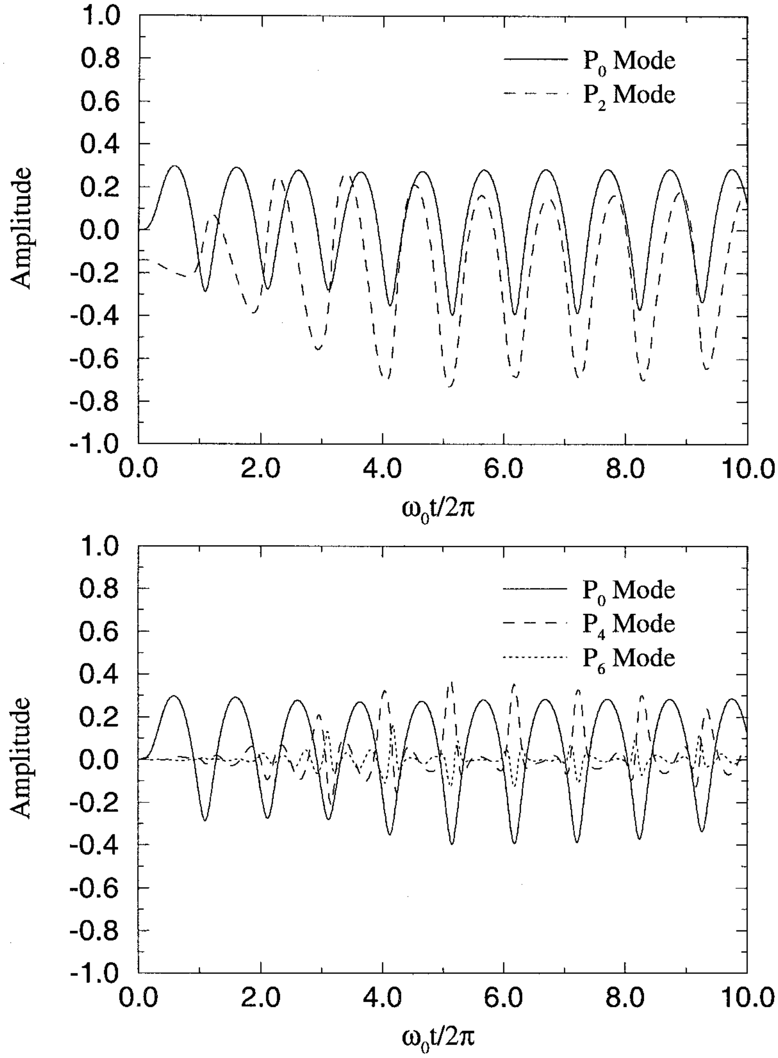

Fig. 5. Amplitudes of the P_0 and P_2 modes (top) and the P_0, P_4 and P_6 modes (bottom) for $\epsilon\beta^* = -0.21$.

modulated and the shape mode amplitudes eventually decrease again as the phase difference between the radial and P_2 mode grows.

The finite amplitude interaction between the radial mode and *all* of the active shape modes becomes still stronger as conditions are changed to further decrease $\epsilon\beta^*$ to -0.51, Fig. 6. By comparing this result with Fig. 4, the lack of symmetry with regard to $\epsilon\beta^*$ in the interaction between modes is readily apparent. This observation suggests that the finite amplitude effect that couples radial oscillations to amplification of the higher order shape modes does *not* depend directly upon the proximity to the exact resonant condition, although the 1:1 resonant interaction is presumably responsible for the initial growth of the P_2 mode. The amplification of the shape mode amplitudes during the collapse of the bubble is very clearly seen in Fig. 6. The corresponding decrease in the volume response and the acoustic pressure compared to

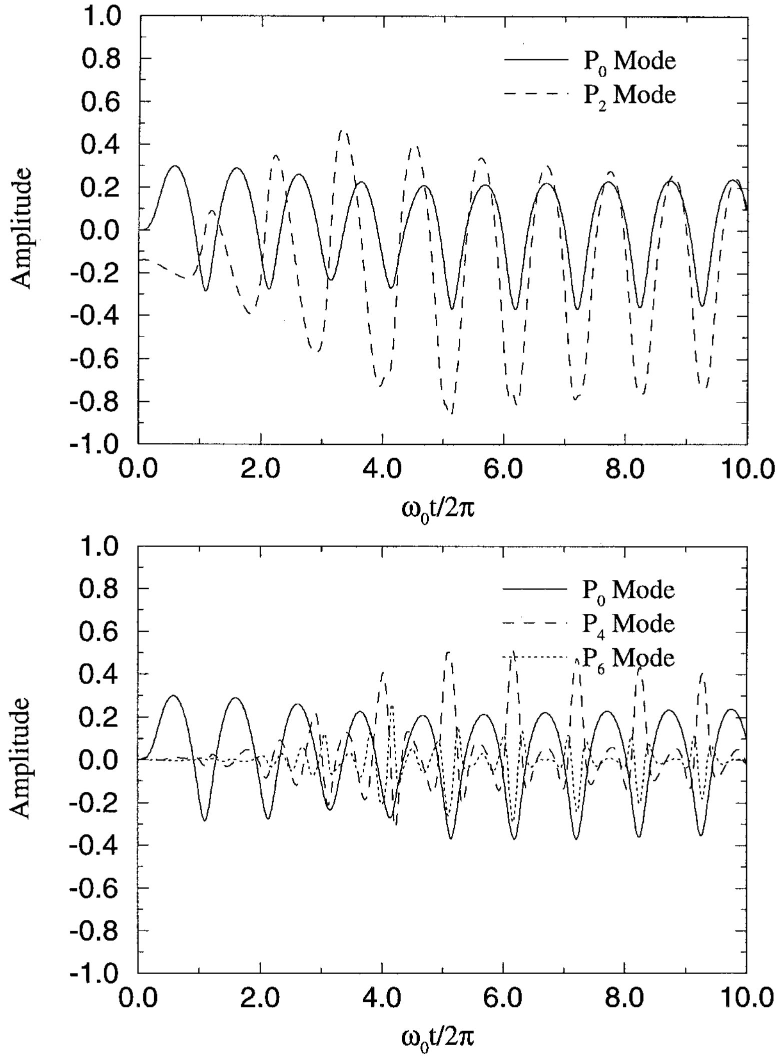

Fig. 6. Amplitudes of the P_0 and P_2 modes (top) and the P_0, P_4 and P_6 modes (bottom) for $\epsilon\beta^* = -0.51$.

predictions of Rayleigh–Plesset theory is illustrated in Fig. 7. The acoustic pressure, in dimensionless variables, is approximately given by

$$P_{\mathrm{A}}(r,t) = \frac{1}{4\pi r}\frac{\mathrm{d}^2 V}{\mathrm{d}t^2} \tag{3}$$

where r is the distance from the center of the bubble (Ceccio and Brennen 1991). Scaling the acoustic pressure by $(2\pi/\omega_0)^2$ in Fig. 7 accounts for using the re-scaled time $\omega_0 t/2\pi$ in the figures and facilitates comparison between results at different equilibrium conditions. The transfer of energy from the radial mode to the shapes modes during the collapse of bubble 'defocuses' the volume collapse, making the bubble less efficient as a monopole source of sound.

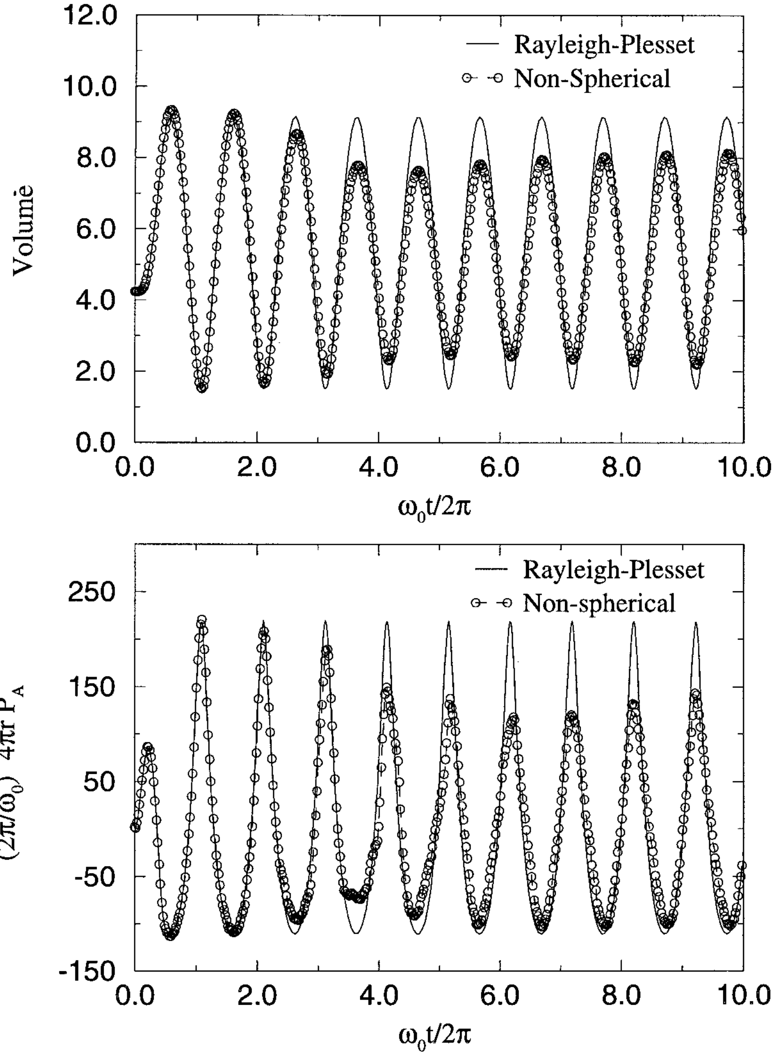

Fig. 7. Comparisons between the volume response (top) and the acoustic pressure (bottom) predicted by Rayleigh–Plesset theory and the numerical simulation of a non-spherical bubble for $\epsilon\beta^* = -0.51$.

To further examine the interaction between large amplitude radial oscillations and oscillation of the shape modes we consider the case with $\epsilon\beta^* = -0.71$ in Fig. 8. We choose to illustrate this case because it shows the *greatest deviation* from the predictions of the Rayleigh–Plesset theory, as shown in Fig. 9. The finite amplitude interaction between oscillations of the radial mode and the shape modes is a consequence of the fact that the interface is a closed surface. As the bubble expands, the area of the surface increases and the amplitudes of waves on the surface are decreased and their wavelength is increased. During the collapse of the bubble the amplitudes of waves on the surface increase as the area of the interface shrinks. This geometric amplification of disturbances on the bubble surface is responsible for the fact that collapsing

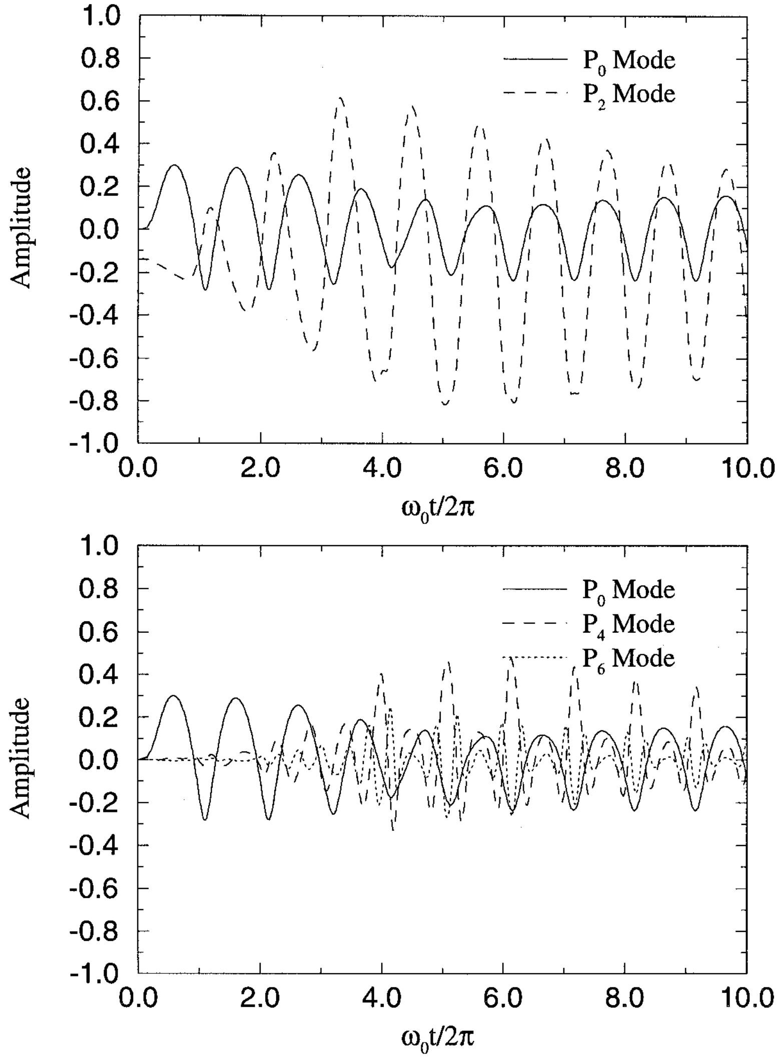

Fig. 8. Amplitudes of the P_0 and P_2 modes (top) and the P_0, P_4 and P_6 modes (bottom) for $\epsilon\beta^* = -0.71$.

bubbles are observed to be unstable (and often break up into a bubble cloud), even though the dense external fluid is accelerated toward the light internal fluid during the collapse phase, a situation which Rayleigh–Taylor (1950) analysis would predict to be stable. It was demonstrated by both Birkhoff (1954, 1956) and Plesset and Mitchell (1956) that in the limit of the radial mode amplitude decreasing to zero, the shape mode amplitudes should grow in proportion to the radial mode amplitude raised to −0.25 power. In terms of the variables this result can be expressed as

$$\epsilon\alpha_{1,n} \propto (\epsilon\alpha_{1,0})^{-0.25} \tag{4}$$

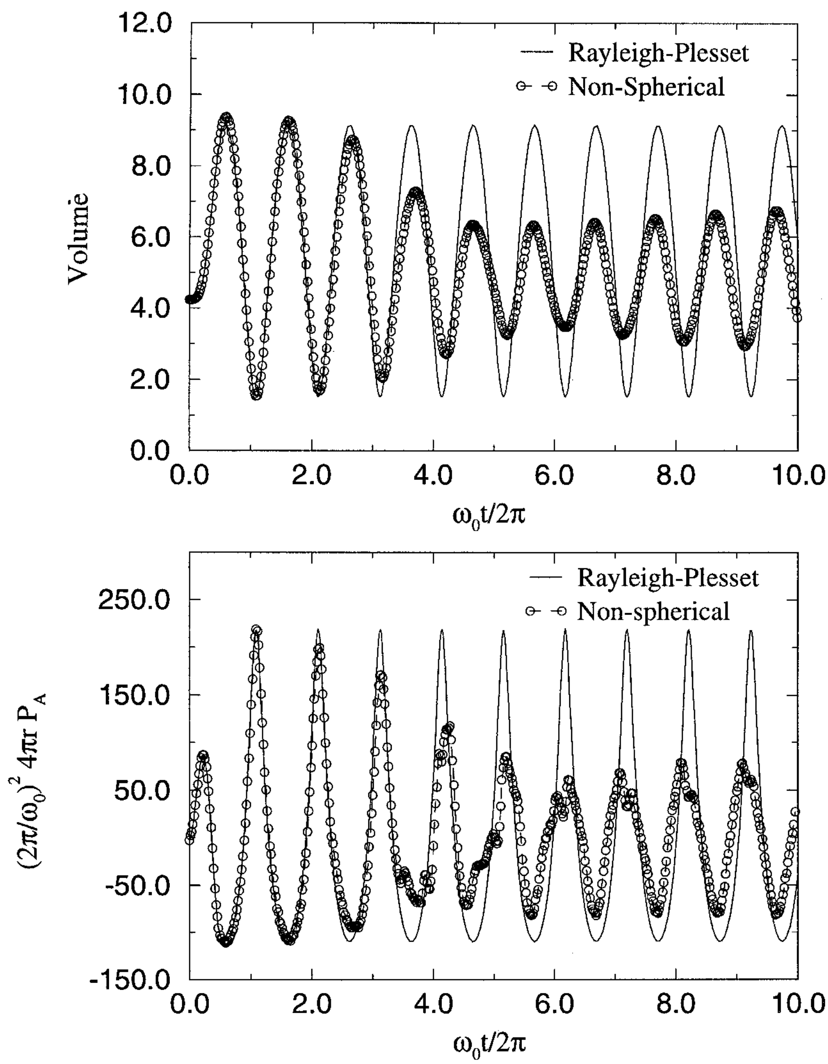

Fig. 9. Comparisons between the volume response (top) and the acoustic pressure (bottom) predicted by Rayleigh–Plesset theory and the numerical simulation of a non-spherical bubble for $\epsilon\beta^* = -0.71$.

In Fig. 10 we construct a log–log plot of the amplitudes for several shape modes against the amplitude of the radial mode during the sixth collapse of the bubble in Fig. 8 and compare our results with the relation (4). While we are clearly *not* in the limit of $\epsilon\alpha_{1,0} \rightarrow 0$, the P_2 mode amplitude does approximately exhibit the power law behavior expected from Eq. (4). The collapse does not proceed to small enough radial mode amplitude to make any judgment about the asymptotic behavior of the higher modes. Still, this result is evidence that the finite amplitude effect seen numerically is a consequence of a geometric amplification of the shape mode amplitudes. The evolution of the bubble shape during the collapse is shown in Fig. 11. The first shape, the large prolate ellipse, is from $\omega_0 t/2\pi = 5.72$ and the final shape, the small oblate shape, is from $\omega_0 t/2\pi = 6.16$.

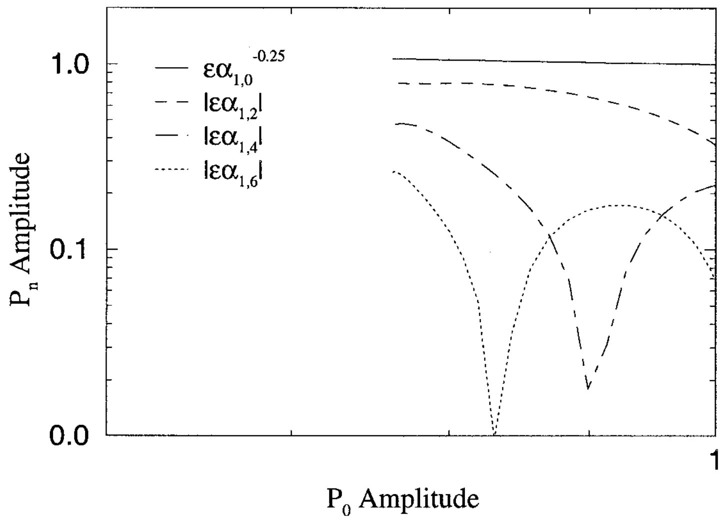

Fig. 10. Log–log plot of the amplitudes of several shape modes versus the amplitude of the radial mode during the sixth collapse in Fig. 8.

All of the results presented thus far are similar in that the radial mode oscillations and the P_2 mode oscillations are in phase when the geometric amplification of the higher-order shape mode amplitudes occurs. The previous analytic and numerical work suggests that when the natural frequency for radial oscillations is less than the natural frequency for P_2 mode oscillations, the radial and P_2 mode oscillations will be π radians out of phase as they interact. In the results presented so far, the frequencies of the radial and/or shape modes are shifted due to finite amplitude oscillation effects so that their oscillations remain in phase. This phase-locking is responsible for the loss of symmetry with regard to $\epsilon\beta^*$ in the interaction between the modes. On the one hand, this is understandable as both the P_0 and P_2 modes are

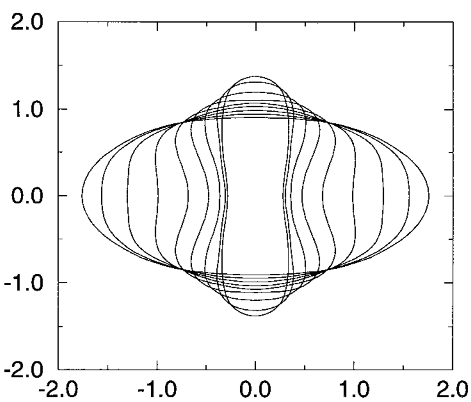

Fig. 11. The evolution of bubble shapes during the sixth collapse in Fig. 8. The full bubble shape is obtained by revolution around the horizontal axis.

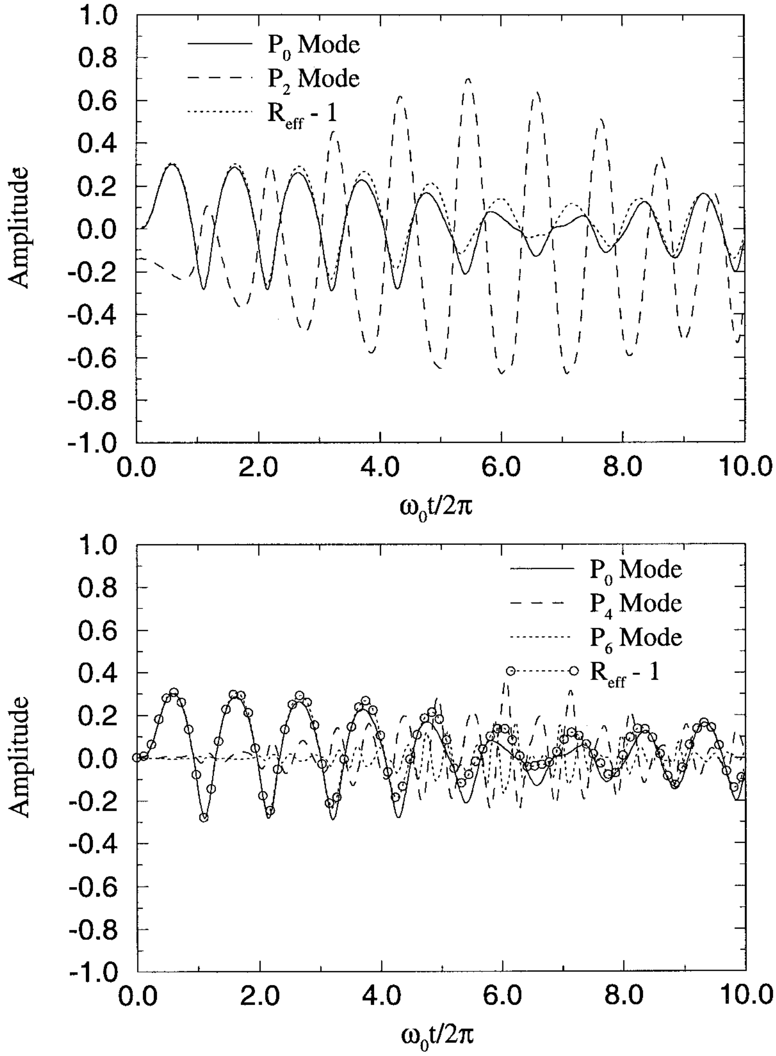

Fig. 12. Amplitudes of the P_0 and P_2 modes (top) and the P_0, P_4 and P_6 modes (bottom) for $\epsilon\beta^* = -0.91$.

significant in determining the volume of the bubble and their amplitudes appear in the expression for the volume with the same sign. Therefore, the in-phase growth in the P_2 mode as the bubble collapses further decreases the bubble volume. However, at some point the natural frequency for radial oscillations will be sufficiently below the natural frequency for P_2 oscillations, that little or no interaction between the modes will occur. The shape modes will not be excited on the bubble surface and no finite amplitude coupling will take place. For conditions which lie *between* these cases we expect to find results where the P_0 and P_2 modes will be π radians out of phase, based upon the 1:1 resonance results from Part I.

In Fig. 12, under conditions where $\epsilon\beta^* = -0.91$, we present such a result. As the bubble collapses the geometric amplification effect causes the P_2 mode amplitude to grow rapidly, but

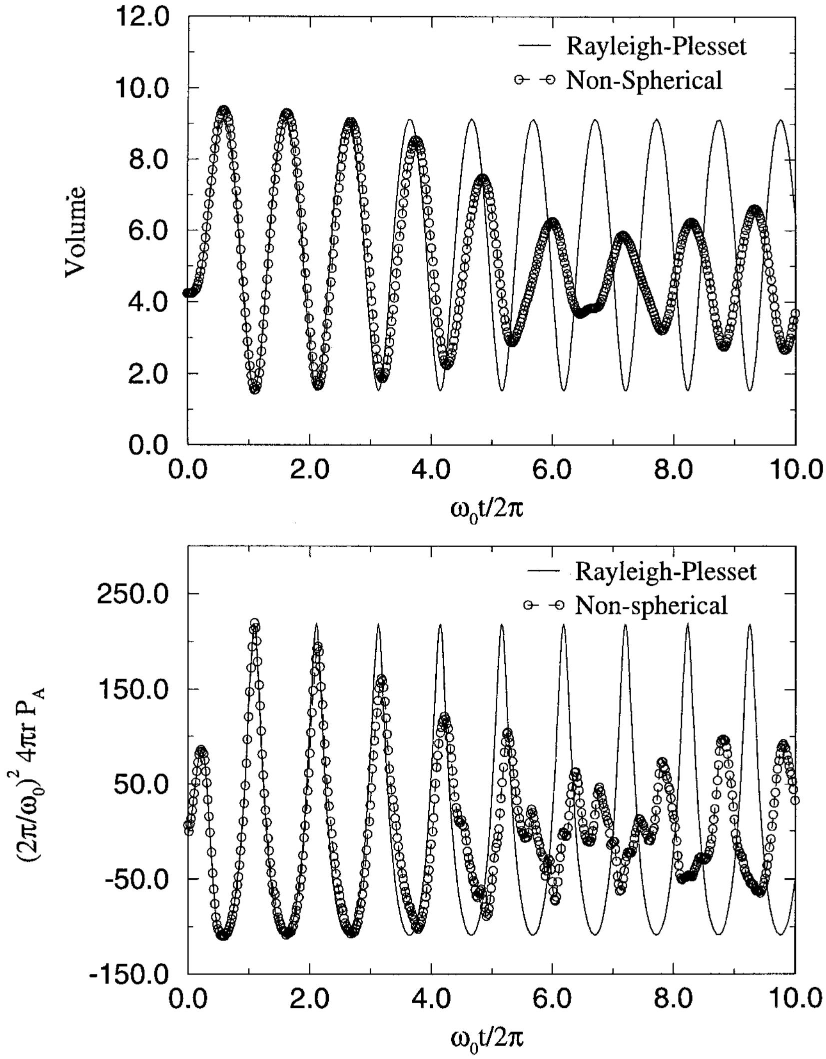

Fig. 13. Comparisons between the volume response (top) and the acoustic pressure (bottom) predicted by Rayleigh–Plesset theory and the numerical simulation of a non-spherical bubble for $\epsilon\beta^* = -0.91$.

the phase difference of π radians means that the P_2 amplitude has the opposite sign of the P_0 mode amplitude and will *add* to the volume of the bubble. As a result, the volume response of the bubble will be *inhibited* and the geometric amplification of the higher-order shape mode amplitudes becomes less apparent. We have added the effective radius, $R_{\text{eff}} = \sqrt[3]{3V/4\pi}$, to Fig. 12 as a measure of the volume response. The result is compared to the volume response predicted by Rayleigh–Plesset theory in Fig. 13. The range of conditions characterized by this second type of a response is very small. In fact, as we move to the final result in this series with $\epsilon\beta^* = -1.01$ the interaction between modes abruptly disappears as the 1:1 resonant interaction no longer leads to shape oscillations of significant amplitude, Fig. 14.

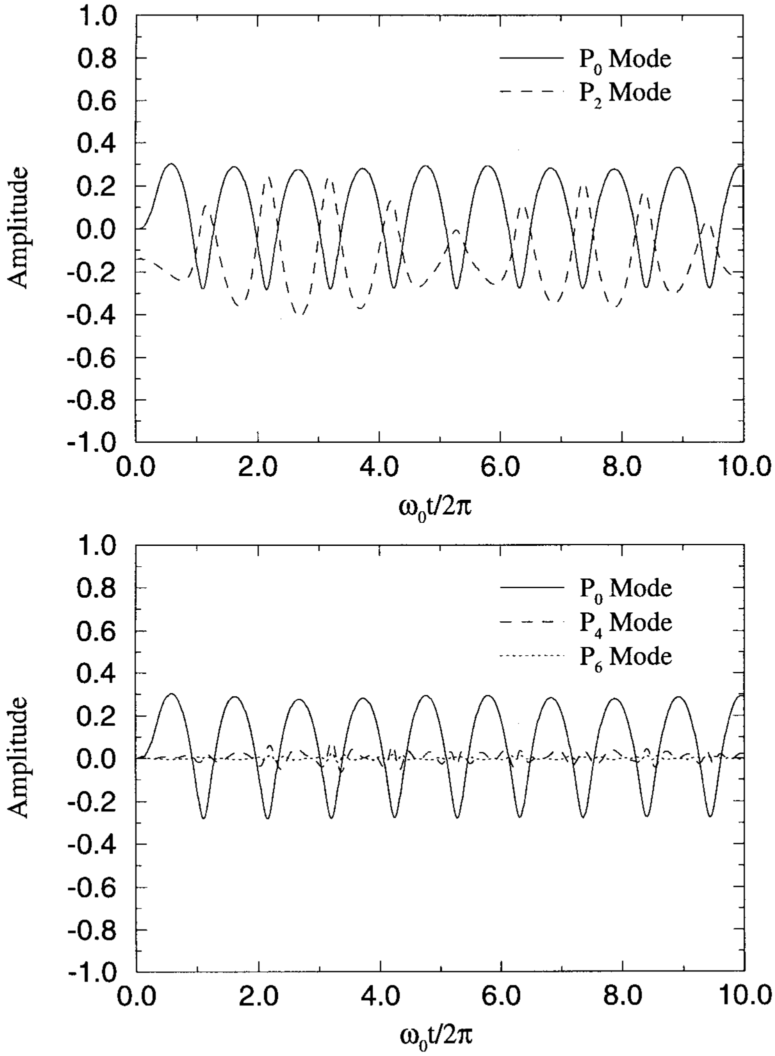

Fig. 14. Amplitudes of the P_0 and P_2 modes (top) and the P_0, P_4 and P_6 modes (bottom) for $\epsilon\beta^* = -1.01$.

The preceding sequence of results demonstrate finite amplitude effects which reduce the efficiency of a bubble as a source of sound. The most common result is that the radial and P_2 mode oscillations become locked in-phase and energy is transferred to *all* of the shape modes during the collapse of the bubble, which defocuses the collapse and reduces both the volume response and the acoustic response of the bubble. For a small set of equilibrium conditions, however, the interaction between the radial mode and the P_2 mode dominates the volumetric response of bubble. For those cases, which occur for a narrow range of conditions in which $\omega_0 < \omega_2$, the P_0 and P_2 modes become locked at π radians out of phase, the geometric amplification of the P_2 mode amplitude then increases the bubble volume, and directly inhibits the collapse of the bubble volume.

The significance of the proximity to the exact 1:1 resonance condition lies in the observation that finite amplitude effects *do not* cause the interface to be unstable, at least for the amplitude of disturbance considered in this study. The response of an *initially spherical* bubble to the same impulsive decrease in pressure will be spherical oscillations, which are therefore identical to what would be predicted by Rayleigh–Plesset theory. Therefore, the finite amplitude amplification of the shape mode amplitudes is only seen when shape modes become active in the bubble response via 1:1 resonance, for P_2 mode oscillations, or by modal coupling for the higher modes. However, the mean deformation and 1:1 resonance are not necessary for the finite amplitude effects to be observed if there is some other source for P_2. For example, if an

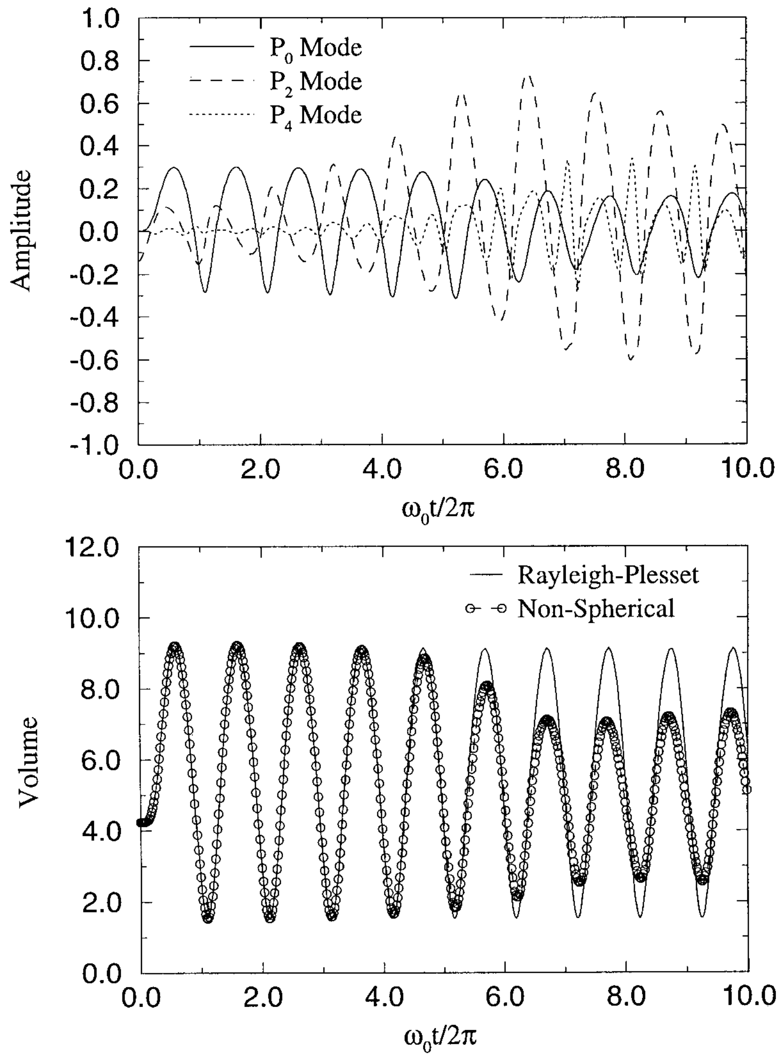

Fig. 15. Amplitudes of the P_0, P_2 and P_4 modes (top) and the comparison of the volume response with the Rayleigh–Plesset theory (bottom) for $A_2 = 0$, $\epsilon\beta^* = -0.91$.

initial perturbation of the P_2 mode is put in at $t=0$ for a bubble with a *spherical* mean shape (i.e. $A_2=0$), the finite amplitude interaction between the modes is still observed following an impulsive decrease in pressure at $t=0$. Fig. 15 shows the response of a bubble under these conditions. One interesting point here is that 1:1 resonance does not occur without the presence of a mean deformation. Thus, in the result of Fig. 15, the phase difference between the radial mode and P_2 shape mode does not get locked at π radians Instead, the finite amplitude interaction between modes is delayed until the P_0 and P_2 modes are in phase with each other. In fact, without the mean deformation, and the accompanying 1:1 resonant interaction, the P_0 and P_2 modes do not get *locked* in-phase as they did for cases with $-0.71 < \epsilon\beta^* < 0.29$. Geometric amplification of the shape modes still occurs as the bubble collapses, but its effect on the volume response of the bubble is greatest when the P_0 mode and P_2 mode have a phase difference of either 0 or π radians.

The finite amplitude effect can also be important in bubble breakup. The amplification of the

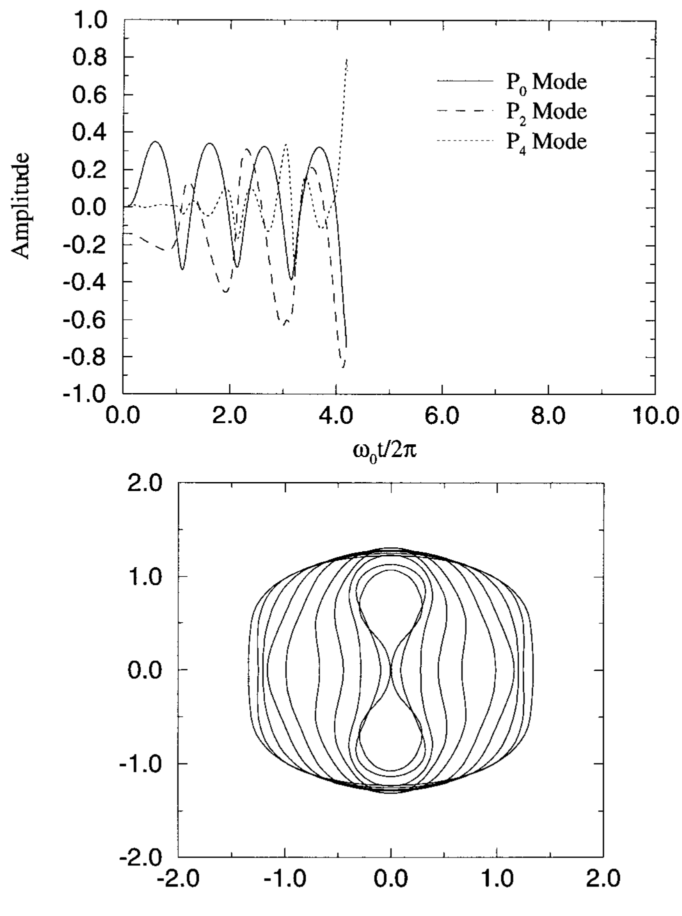

Fig. 16. Amplitudes of the P_0, P_2 and P_4 modes (top) and the evolution of the bubble shapes during the final collapse (bottom) for $\epsilon\beta = -0.21$.

shape mode amplitudes as the bubble collapses is precisely what is required to drive the bubble to breakup in the *absence* of strong spatial gradients in pressure or stress. If we consider an example with a slightly stronger impulsive pressure decrease that gives an initial radial expansion of $\varepsilon\alpha_{1,0} = 0.35$, we find that bubble breakup occurs for all cases with $-0.41 < \epsilon\beta^* < -0.21$. A representative case is shown in Fig. 16. The bubble breaks via the formation of a toroidal bubble in which opposing surfaces of the bubble approach each other along the axis of rotational symmetry. The subsequent evolution, and presumably breakup, of the toroidal bubble is beyond the scope of our investigation. However, if the toroidal bubble were to break up via a capillary wave instability to form several smaller bubbles, this mechanism for bubble breakup may be an important step in bubble cloud formation.

4. Summary of results

- Non-resonant finite amplitude interactions between radial and shape mode oscillations result in rapid increase in the amplitude of shape oscillations during bubble collapse, cf. Figs. 2, 4–6, 8 and 12.
- Finite amplitude interactions between radial and shape mode oscillations reduce the efficiency of a bubble as a monopole source of sound by transferring energy into higher order shape modes and 'defocusing' the bubble collapse, Figs. 7, 9 and 13.
- The finite amplitude interaction is a consequence of a geometric amplification of shape mode amplitude, Fig. 10 and accompanying discussion.
- The geometric amplification of shape modes is independent of the presence of a mean deformation and the proximity to resonance conditions, Fig. 15.
- 1:1 resonant coupling between radial and shape oscillations provides a mechanism to excite shape modes in the bubble response that subsequently grow in amplitude via geometric amplification, cf. Figs. 2, 4–6, 8 and 12. The resonant coupling also affects the phase relationship between radial and shape mode oscillations cf. Figs. 8 and 12.
- Geometric amplification of shape modes can also be important in bubble breakup, Fig. 16.

5. Discussion

We have shown in the present paper (Part II) that the efficiency of a bubble as a monopole source of sound can be dramatically decreased when finite amplitude interactions between the radial mode oscillations and the oscillations of P_n shape modes leads to strong growth in the amplitudes of higher-order modes (larger n). The geometric amplification of the amplitude of shape modes, which plays an essential role in this phenomenon, is *not* dependent on the presence of a mean deformation, nor directly on proximity to resonance conditions. However, it does require the presence of shape modes of finite amplitude, and the 1:1 resonant interaction between P_0 and P_n oscillations (P_2 in the present case) can play an important role in exciting shape mode oscillations, especially if the initial perturbation is strictly radial. We have also seen that proximity to resonance conditions does play a role in determining the relative phase between the oscillations of the radial mode and the P_2 mode (for our case, it is the P_2 mode that is *directly* affected by resonant coupling), and this plays an important secondary role in determining whether the amplitude of volume oscillations is sufficient to drive the coupling to higher order modes.

One apparent difference between the present results and the experimental observation of an acoustic signal in the traveling bubble hydroacoustics problem is that we do not typically predict large effects on the bubble volume until after several periods of oscillation have already gone by. In the computations, however, the initial perturbation of shape is a relatively small P_2 contribution, and it takes several periods before this can lead, via mode coupling effects, to higher-order modes of sufficient amplitude to produce a major influence on the magnitude of volume oscillations. In the real problem, the initial shape is much more complex, presumably containing the higher-order shape modes at finite amplitude immediately.

We have seen, in the two parts of the present investigation, that the dynamics of non-spherical bubbles is generally quite different than predictions based on the Rayleigh–Plesset theory for spherical bubbles. Specifically, when the equilibrium or steady-state conditions are such that we are in the neighborhood of resonant interactions between the purely radial oscillation and one of the P_n shape modes, there is strong coupling between the shape and volume oscillations and this leads to a decrease in the amplitude of the oscillatory changes in bubble volume. For very small amplitude oscillations, analytic theory and numerical simulations both show that the coupling is limited to the radial mode and a single shape mode, which interact as a non-linear oscillator. In this regime, provided the departures of shape and volume from equilibrium or steady-state conditions are small enough, the analytic theory provides a basis for predicting the effect of shape deformation on the acoustic output of an oscillating bubble.

For larger amplitude oscillations, the small amplitude theory still provides some qualitative guidance (for example, it provides a reasonable estimate of the critical frequency detuning for instability of a purely radial oscillation via 2:1 resonant coupling), but it is generally unable to provide quantitatively correct results, and also fails to capture some of the most important qualitative features. Foremost among these is the role of energy transfer to higher-order modes in decreasing the magnitude of volumetric oscillations, and especially the geometric amplification effect that is evident in the present simulations (Part II) of the bubble response to an initial impulsive decrease in the pressure within the surrounding fluid.

An obvious question is how the acoustics engineer is supposed to estimate the sound due to bubbles. In current practice, this is generally done by using a measured or predicted pressure history as input to the Rayleigh–Plesset theory. The message from our present study, as limited in scope as it may be, is that deviations of the bubble shape from spherical can lead to a major reduction of the strength of the bubble as an acoustic source relative to Rayleigh–Plesset predictions. Unfortunately, however, the *analytic* small amplitude theory, which reduces the problem to interactions with a single P_n shape mode, is too simple, and does not provide a viable *alternative* basis for acoustics predictions except for exceedingly small amplitude effects. On the other hand, the complexity of simulating the full dynamics of a bubble of complicated shape is overwhelming as a tool for effective prediction of bubble acoustics. What is needed is a global modal decomposition, of the type currently being developed for the reduction of turbulent flow dynamics ('proper orthogonal decomposition'), to allow an approximate description of the full dynamics via a relatively low order dynamical system. Unfortunately, such a decomposition, even if possible, requires a great deal of 'data' on the dynamics of bubbles of complex shape, which does not presently exist.

Acknowledgements

This research was supported via grants from ONR and NASA. NKM was also supported by a doctoral fellowship from NASA. The authors gratefully acknowledge this support.

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